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Multiparameter spectral theory and separation of variables

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Abstract

Multiparameter spectral theory generalizes the classical spectral theory of linear operators to n linear operators linked by n spectral parameters. The spectral parameters arise as separation constants when, for example, separation of variables techniques are used to solve boundary-value problems for partial differential equations. In this paper, we discuss some of the fundamental aspects of the theory such as solvability, commutativity and abstract representation theory. Many of these concepts are of importance to modern developments in separation of variables techniques. The ideas are illustrated in application to differential equations and special functions of mathematical physics.

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In memory of Vadim B Kuznetsov 1963–2005.

1. Introduction

Multiparameter spectral theory has its roots in the classic problem of solving boundary-value problems for partial differential equations by classical methods of separation of variables. In the standard case, the separation technique leads to the study of systems of ordinary differential equations linked by spectral parameters (i.e. separation constants) in an elementary way. For example, the problem of small oscillations of a rectangular membrane with fixed boundary leads to a pair of Sturm–Liouville eigenvalue problems which are separated not only as regards their independent variables but also in regards to the spectral parameters as well. The same problem posed for the circular membrane leads to only mild parametric linking. This is a kind of triangular situation. The parameter in the angular equation must be determined by periodicity and the resulting values substituted into the radial equation leading to the study of various Bessel functions. The full multiparameter situation arises in full if we pursue this class of problems a little further. Consider for example the vibration problem of an elliptic membrane with clamped boundary. It is appropriate here to use elliptic coordinates. Application of the separation of variables method leads to the study of eigenvalue problems for a pair of ordinary differential equations both of which contain the same two spectral parameters. This is then a genuine two-parameter problem. The ordinary differential equations which arise are Mathieu equations with solutions expressible in terms of Mathieu functions. Other problems of this type give rise to two- or three-parameter eigenvalue problems and their resolution lies to a large extent in the properties of the 'higher' special functions of mathematical physics, e.g. Lamé functions, spheroidal wavefunctions, paraboloidal wavefunctions, ellipsoidal wavefunctions, etc. Many of these special functions have been vigorously studied over the past century. In contrast, the general case of multiparameter spectral theory has been rather neglected over the years despite the fact that it arose almost as long ago as the classic work of Sturm and Liouville on one-parameter eigenvalue problems.

In its most general setting, the multiparameter eigenvalue problem for ordinary differential equations may be formulated in the following manner.

Consider the system of n ordinary, second-order, linear, formally self-adjoint differential equations in the n parameters, $\lambda_1, \ldots, \lambda_n$, $n \geqslant 2$,

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}x_r^2} + \left\{ \sum_{s=1}^{n} a_{rs}(x_r)\lambda_s - q_r(x_r) \right\} y_r = 0, \tag{1.1}$$

$0 \leqslant x_r \leqslant 1, r = 1, \ldots, n$ with $a_{rs}(x_r), q_r(x_r)$ continuous and real-valued functions. By writing λ for $(\lambda_1, \ldots, \lambda_n)$ we may formulate an eigenvalue problem for (1.1) by demanding that λ be chosen so that equations (1.1) have non-trivial solutions with each satisfying the homogeneous boundary conditions:

$$\begin{aligned} &\cos\alpha_r\, y_r(0) - \sin\alpha_r \frac{\mathrm{d}y_r(0)}{\mathrm{d}x_r} = 0, \qquad 0 \leqslant \alpha_r < \pi, \\ &\cos\beta_r\, y_r(1) - \sin\beta_r \frac{\mathrm{d}y_r(1)}{\mathrm{d}x_r} = 0, \qquad 0 < \beta_r \leqslant \pi, \\ &r = 1, \ldots, n. \end{aligned} \tag{1.2}$$

If λ can be so chosen, it is called an eigenvalue and the corresponding product $\prod_{r=1}^{n} y_r(x_r, \lambda)$ is called the eigenfunction.

Much of the early work regarding the systems (1.1) and (1.2) was concerned with certain extensions of the Sturmian oscillation theory. As regards spectral theory and in particular questions related to the completeness of eigenfunctions and the like some further structure must be imposed on the system. It is clear from the formulation that a tensor product space setting is necessary. Furthermore, we need multiparameter analogues of the classical 'right' and 'left' definiteness positivity conditions essential to one-parameter spectral theory.

For the multiparameter eigenvalue problems (1.1) and (1.2), the appropriate positivity conditions are

$$\text{(A)} \quad \Delta_n \equiv \det\{a_{rs}\}_{r,s=1}^{n} > 0, \tag{1.3}$$

for all $x = (x_1, \ldots, x_n) \in I^n$ (the Cartesian product of the n intervals, $0 \leqslant x_r \leqslant 1, r = 1, \ldots, n$) or

$$\text{(B)} \quad \Delta_n \neq 0,$$

with

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{r-1,1} & \cdots & a_{r-1,n} \\ \mu_1 & \cdots & \mu_n \\ a_{r+1,1} & \cdots & a_{r+1,n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0, \tag{1.4}$$

for each $r = 1, \ldots, n$ and for some non-trivial real numbers $\mu_1, \ldots, \mu_n$.

The problem defined by (1.1), (1.2) and condition (A) is called a 'right definite' multiparameter eigenvalue problem, whereas the problem defined by (1.1), (1.2) and condition (B) is called 'left definite'.

The structural conditions (A) and (B) have an extension to analogous conditions for abstract linear operators in Hilbert space and these extended conditions are crucial to the development of abstract multiparameter spectral theory.

Since the 1930s, apart from the continued interest in special functions, multiparameter spectral theory remained somewhat neglected until the 1960s when F V Atkinson gave an address to the American Mathematical Society in Iowa City on 27th November 1965. Atkinson's lecture was monumental in that it sets out a wide ranging programme not only including differential equations, but also matrices and matrix pencils, a functional calculus, questions of duality and much more. The lecture was published in 1967 as an expository article in the Bulletin of the American Mathematical Society [3]. Following this, Atkinson brought together his ideas in the finite-dimensional case in the book [4]. A subsequent book dealing with infinite-dimensional problems and applications to differential equations was planned but unfortunately was never published. In 1978, the author published the monograph [26] in which many aspects of the infinite-dimensional case in a Hilbert space setting were discussed. Other works which give excellent and extensive coverage of multiparameter spectral problems are those of Faierman [11], McGhee and Picard [23] and Volkmer [29].

In this paper, we discuss those aspects of multiparameter spectral theory which have important connections with the modern developments in separation of variables as considered by Sklyanin [27] and Kuznetsov *et al* [20]. In section 2, we set down a formal definition of the abstract multiparameter spectral problem and address certain fundamental notions such as solvability of systems of linear operator equations and associated commutativity properties. The expansion and the multiparameter spectral representation theorems are considered in section 3. In section 4, we consider a novel and fundamental abstract relation which throws new light on integral representation results for special functions and also appears to be related in a fundamental way to the factorization techniques of Kuznetsov *et al* [20]. Section 5 outlines a new approach to the factorization problem for Schur polynomials by exploiting the theory of section 4. The paper concludes, in section 6, with a discussion of open problems and suggests new research directions. Throughout we illustrate the ideas with examples and applications, many of which are new to the literature. The time is opportune to concentrate new efforts to understand and develop multiparameter theory in the light of new applications emerging in mathematical analysis and theoretical physics.

2. Commutativity and solvability

In this section, we set down certain fundamental notions regarding the abstract multiparameter eigenvalue problem in Hilbert space.

Consider the abstract problem

$$A_i u^i = \sum_{j=1}^{n} \lambda_j S_{ij} u^i, \qquad i = 1, \ldots, n, \tag{2.1}$$

where the S_{ij} are bounded symmetric operators in a Hilbert space H_i and the A_i are self-adjoint operators in H_i. As in the case of ordinary differential equations, the n-tuple $(\lambda_1, \ldots, \lambda_n)$ of complex numbers is called an eigenvalue of (2.1) if each equation has a non-trivial solution $u^i \in H_i$ for this n-tuple. The corresponding element $u^1 \otimes \cdots \otimes u^n$ in the tensor product space $H = \otimes_{i=1}^{n} H_i$ is called an eigenvector to this eigenvalue. In analogy with the right definiteness condition (A) above, we suppose

$$\det_{1\leqslant i,j\leqslant n} \{(S_{i,j} u^i, u^i)_i\} > 0 \tag{2.2}$$

for all $u^i \neq 0$ in H_i (here $(\cdot, \cdot)_i$ denotes the inner product in H_i). It is then standard to prove that the eigenvalues of (2.1) are real. With the further assumption that each A_i has compact resolvent and that zero is in the resolvent set, Browne [6] established the first completeness theorem for (1.2) under the assumption that

$$A_i^+ f = \sum_{j=1}^{n} S_{ij}^+ g_j, \qquad i = 1, \ldots, n \tag{2.3}$$

is uniquely solvable in $g_1, \ldots, g_n \in H$ for each $f = f^1 \otimes \cdots \otimes f^n$ with $f^i \in D(A_i) \subset H_i$. The operators A_i^+ and S_{ij}^+ denote the operators induced in the tensor product space H by A_i and S_{ij}, respectively. The question of when (2.3) is solvable is fundamental to multiparameter spectral theory and had already been raised by Atkinson [3]. The first result on solvability is due to Källström and Sleeman [17]. To describe this result we begin with a few observations.

Every operator S_{ij} in H_i has a corresponding induced operator S_{ij}^+ in $H = \otimes_{i=1}^{n} H_i$ defined as follows:

$$S_{ij}^+ (u^1 \otimes \cdots \otimes u^n) = u^1 \otimes \cdots \otimes u^{i-1} \otimes S_{ij} u^i \otimes u^{i+1} \otimes \cdots \otimes u^n$$

on separable elements and then extended by linearity and continuity. It is easy to verify that S_{ij}^+ is bounded and symmetric on H and

$$\left\| S_{ij}^+ u \right\| \leqslant \|S_{ij}\|_i \|u\|,$$

where $\|\cdot\|$ denotes the tensor product norm in H and $\|\cdot\|_i$ is the operator norm of S_{ij} in H_i. Since S_{ij} and S_{kl} operate in different spaces when $i \neq k$, the corresponding operators S_{ij}^+ and S_{kl}^+ in H will commute for all choices of j and l, $1 \leqslant j, l \leqslant n$. It then follows that we can define the determinantal operator

$$S = \det \left(S_{ij}^+\right)_{1\leqslant i,j\leqslant n}$$

which is bounded and symmetric on H. We now strengthen condition (2.2) with the following assumption.

Assumption. S is a positive definite operator in H, i.e. there is a constant $C > 0$ such that

$$(Su, u) \geqslant C\|u\|^2, \qquad u \in H. \tag{2.4}$$

This implies in particular that S has a bounded inverse defined on H.

From now on, in this section, all operators will be considered as acting in H and the + notion will be omitted.

By expanding $\det(S_{ij})$ we see that

$$S = \sum_{i=1}^{n} S_{ik}\hat{S}_{ik} = \sum_{i=1}^{n} \hat{S}_{ik}S_{ik}, \qquad k = 1, \ldots, n,$$

where $\hat{S}_{ik}$ is the cofactor of S_{ik} in S. Similarly, we find

$$\sum_{i=1}^{n} S_{ij}\hat{S}_{ik} = S\delta_{jk}, \qquad j, k = 1, \ldots, n, \tag{2.5}$$

where δ_{jk} is the Kronecker delta. Note, however, that if $i \neq k$ then

$$\sum_{j=1}^{n} S_{ij}\hat{S}_{kj} \neq S\delta_{ik}.$$

Consider the problem, find solutions $u_1, \ldots, u_n$ in H of the linear operator system

$$\sum_{j=1}^{n} S_{ij}u_j = f_i, \qquad i = 1, \ldots, n, \tag{2.6}$$

for given $f_1, \ldots, f_n$ in H. Observe that if a solution exists then it is unique and can be constructed by applying $\hat{S}_{ik}$ to the ith equation in (2.6) and summing over i to get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{S}_{ik}S_{ij}u_j = \sum_{i=1}^{n} \hat{S}_{ik}f_i$$

which reduces to

$$u_k = S^{-1}\sum_{i=1}^{n} \hat{S}_{ik}f_i \tag{2.7}$$

for $k = 1, \ldots, n$. This is the usual Cramer's rule solution. However, in order to verify that (2.7) solves (2.6), mere substitution leads to sums of the form

$$\sum_{j=1}^{n} S_{ij}S^{-1}\hat{S}_{kj}$$

which cannot be reduced to simpler terms. So to establish the existence an inductive argument is used. This argument makes use of the following lemma.

Lemma 1. *If S is positive definite on H, there exists a linear combination of cofactors*

$$\sum_{k=1}^{n} \alpha_k\hat{S}_{jk} \quad \text{some} \quad j = 1, \ldots, n,$$

which is positive definite on H. With this basic lemma and a variant of Gaussian elimination, an induction argument proves that (2.6) has a unique solution. For details, see [17].

As an example, consider the case of $n = 2$ and seek the solution of

$$\begin{aligned} S_{11}u_1 + S_{12}u_2 &= f_1, \\ S_{21}u_1 + S_{22}u_2 &= f_2, \end{aligned}$$

where $S = S_{11}S_{22} - S_{12}S_{21}$ is positive definite. Furthermore, an affine transformation and lemma 1 allows us to assume that $S_{11} \equiv \hat{S_{22}}$ is also positive definite.

From the first equation

$$u_1 = S_{11}^{-1}(f_1 - S_{12}u_2),$$

and this inserted in the second equation gives

$$\left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right)u_2 = f_2 - S_{21}S_{11}^{-1}f_1,$$

and by commutativity this reduces to

$$S_{11}^{-1}Su_2 = f_2 - S_{11}^{-1}S_{21}f_1$$

and so

$$u_2 = S^{-1}(S_{11}f_2 - S_{21}f_1).$$

A by-product of the method of solution is that certain commutativity relations hold between the elements of S. The construction above shows that a solution is

$$\begin{aligned} u_1 &= \left(S_{11}^{-1} + S_{11}^{-1}S_{12}S^{-1}S_{21}\right)f_1 - S_{11}^{-1}S_{12}S^{-1}S_{11}f_2, \\ u_2 &= S^{-1}(S_{11}f_2 - S_{21}f_1). \end{aligned}$$

However, we know that the solution exists and is given by Cramer's rule as

$$\begin{aligned} u_1 &= S^{-1}(S_{22}f_1 - S_{12}f_2), \\ u_2 &= S^{-1}(S_{11}f_2 - S_{21}f_1). \end{aligned}$$

Finally, on comparing these forms of solutions gives the following identities:

$$\begin{aligned} &S_{11}S^{-1}S_{22} - S_{12}S^{-1}S_{21} = I, \\ &S_{12}S^{-1}S_{11} = S_{11}S^{-1}S_{12}. \end{aligned} \tag{2.8}$$

In addition, a careful study of the methods of proof used to establish (2.8) shows that it holds without requiring S_{11} to be positive definite and in addition it follows that

$$S_{21}S^{-1}S_{22} = S_{22}S^{-1}S_{21}. \tag{2.9}$$

As a consequence, we see that the operators $S^{-1/2}S_{11}S^{-1/2}$ and $S^{-1/2}S_{12}S^{-1/2}$ commute as well as the operators $S^{-1/2}S_{21}S^{-1/2}$ and $S^{-1/2}S_{22}S^{-1/2}$. Similar but more complicated identities hold in the general $n \times n$ case.

The fundamental problem of the solvability of systems of operator equations generalizes in a number of directions which allows one to develop multiparameter spectral theory under more general structural conditions than condition (A) above. To illustrate this, suppose we have operators $A_i, S_{ij}, i, j = 1, \ldots, n$ with the property

(B) $A_i, S_{ij} : H_i \to H_i, i, j = 1, \ldots, n$ are Hermitian and continuous.

In addition, we shall impose a general definiteness condition which is introduced as follows: let $f = f^1 \otimes \cdots \otimes f^n$ be an element of H with $f_i \in H_i, i = 1, \ldots, n$ and let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be a given set of real numbers not all zero. Then the operators $\Delta_i : H \to H, i = 1, \ldots, n$ can be defined by the equation

$$Af = \sum_{i=0}^{n} \alpha_i \Delta_i f = \det \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ -A_1 f_1 & S_{11} f_1 & \cdots & S_{1,n} f_1 \\ \vdots & \vdots & & \vdots \\ -A_n f_n & S_{n1} f_n & \cdots & S_{n,n} f_n \end{vmatrix}, \tag{2.10}$$

where the determinant is to be expanded formally using the tensor product. This defines $\Delta_i f$ for decomposable $f \in H$ and we extend the definition to arbitrary $f \in H$ by linearity and continuity. We can now introduce the definiteness condition

$$\text{(C)} \quad (Af, f) \geqslant C\|f\|^2. \tag{2.11}$$

We now pose the problem, given $f \in H$ find elements $f_i \in H, i = 1, \dots, n$, satisfying the system of equations

$$\begin{aligned} &\sum_{i=0}^{n} \alpha_i f_i = f, \\ &-A_i^+ f_0 + \sum_{j=1}^{n} S_{ij}^+ f_j = 0, \qquad i = 1, \dots, n. \end{aligned} \tag{2.12}$$

The arguments developed in [17] show that (2.12) subject to condition (C) has a unique solution for any $f \in H$ given by Cramer's rule:

$$f_i = (A^+)^{-1}\Delta_i^+ f, \qquad i = 0, \dots, n, \tag{2.13}$$

where the operators $A^+, \Delta_i^+ : H \to H$ are those induced by A, Δ_i defined in (2.10). This time we deduce the following commutativity relations.

Lemma 2. *The operators appearing in the system (2.12) enjoy the following commutativity properties:*

$$\begin{aligned} &\sum_{j=0}^{n} \alpha_j (A^+)^{-1}\hat{\alpha}_j = I, \\ &\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1}\hat{S}_{ij} = I, \qquad i = 1, \dots, n, \\ &\sum_{j=0}^{n} \alpha_j (A^+)^{-1}\hat{S}_{kj} = 0, \qquad k = 1, \dots, n, \\ &\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1}\hat{\alpha}_j = 0, \qquad i = 1, \dots, n, \\ &\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1}\hat{S}_{kj} = 0, \qquad k \neq i, \quad i, k = 1, \dots, n. \end{aligned}$$

For proofs of the above results and further extensions, we refer to Browne and Sleeman [7], Källström and Sleeman [18] and Binding and Košir [5]. Before moving onto the importance of these results to multiparameter spectral theory, we mention that the commutativity relations above have been recently re-discovered independently by Enriquez and Rubtsov [9] in their work on commuting families in skew algebraic fields.

3. Completeness and expansion theory

In this section, we outline the main ideas leading to results which are fundamental to establishing the completeness of eigenfunctions of the multiparameter eigenvalue problem via the Parseval equality and the associated eigenvector expansion. Indeed we present an essentially unified multiparameter spectral theory. For the purposes of presentation, we continue to assume that the operators $A_i, S_{ij}, i, j = 1, \ldots, n$ and A defined by (2.10) satisfy conditions (B) and (C) above. Rather than use the inner product $(\cdot, \cdot)$ in the tensor product space H generated by the inner products $(\cdot, \cdot)_i$ in H_i, we use the inner product given by $(A^+\cdot, \cdot)$ which will be denoted by $[\cdot, \cdot]$. The norms induced by these inner products are equivalent and so topological concepts such as continuity of operators and convergence of sequences may be discussed unambiguously without reference to a particular inner product. Algebraic concepts however may depend on the inner product. For an operator $L : H \to H$ we denote by L^* the adjoint of L with respect to $[\cdot, \cdot]$, i.e. for all $f, g \in H$ we have

$$[Lf, g] = [f, L^*g]. \tag{3.1}$$

For the operators $\Gamma_i : H \to H, i = 0, 1, \ldots, n$ defined by

$$\Gamma_i \equiv (A^+)^{-1}\Delta_i, \tag{3.2}$$

we have the following theorem.

Theorem 3.1. $\Gamma_i = \Gamma_i{}^*$.

The proof of this is an immediate consequence of the definition of adjoint. Working with the inner product $[\cdot, \cdot]$ in H, the operators $\Gamma_i, i = 0, 1, \ldots, n$ form a family of $(n + 1)$ commuting Hermitian operators. Let $\sigma(\Gamma_i)$ denote the spectrum of Γ_i and $\sigma_0 = \times_{0\leqslant i\leqslant n}\sigma(\Gamma_i)$, the Cartesian product of the $\sigma(\Gamma_i), i = 0, 1, \ldots, n$. Then since $\sigma(\Gamma_i)$ is a non-empty compact subset of $\mathbb{R}$, it follows that σ_0 is a non-empty compact subset of $\mathbb{R}^{n+1}$.

Let $E_i(\cdot)$ denote the resolution of the identity for the operator Γ_i and let $M_i \subset \mathbb{R}$ be a Borel set, $i = 0, 1, \ldots, n$. We then define $E(M_0 \times M_1 \times \cdots \times M_n) = \prod_{i=0}^{n} E_i(M_i)$. Note the projections $E_i(\cdot)$ will commute since the operators Γ_i commute. Thus in this way we obtain a spectral measure $E(\cdot)$ on the Borel subsets of $\mathbb{R}^{n+1}$ which vanishes outside σ_0. Measures of the form $[E(\cdot)f, f]$ will be non-negative finite Borel measures vanishing outside σ_0.

The spectrum σ of the system $\{A_i, S_{ij}\}$ may be defined as the support of the operator-valued measure $E(\cdot)$, i.e. σ is the smallest closed set outside of which $E(\cdot)$ vanishes or alternatively σ is the smallest closed set with the property $E(M) = E(M \cap \sigma)$ for all Borel sets $M \subset \mathbb{R}^{n+1}$. Thus, σ is a compact subset of $\mathbb{R}^{n+1}$ and if $\lambda \in \sigma$ then for all non-degenerate closed rectangles M with $\lambda \in M, E(M) \neq 0$. Thus the measures $[E(M)f, g], f, g \in H$ actually vanish outside σ.

We are now in a position to state our main result, namely, the Parseval equality and eigenvector expansion theorem.

Theorem 3.2. *Let $f \in H$. Then*

$$(i) \quad (A^+f, f) = \int_\sigma [E(\mathrm{d}\lambda)f, f] = \int_\sigma (E(\mathrm{d}\lambda)f, A^+f). \tag{3.3}$$

$$(ii) \quad f = \int_\sigma E(\mathrm{d}\lambda)f, \tag{3.4}$$

where this integral converges in the norm of H.

This theorem is an easy consequence of the theory of functions of several commuting Hermitian operators.

4. Abstract relations

In one-parameter spectral theory of ordinary differential operators, certain insights are to be gained by reformulating differential equations as equivalent integral equations. The extension of this idea to the multiparameter case has been considered by Arscott [1]. In the case of the abstract problem (2.1), the analogue of such integral equations is a relation defined on H, the tensor product of the spaces H_i, $i = 1, \ldots, n$.

We begin with some basic notations and concepts.

Let $w = w_1 \otimes \cdots \otimes w_n \in H$, $w_i \in H_i$ be a decomposed element, and for a fixed $u_n \in H_n$ consider the mapping

$$w \to w_1 \otimes \cdots \otimes w_{n-1}(w_n, u_n)_{H_n} \in \otimes_{i=1}^{n-1} H_i,$$

where $(\cdot, \cdot)_{H_n}$ denotes the inner product in H_n. This mapping can conveniently be written as $\langle w, u_n\rangle_{H_n}$ and may be extended from decomposable w to the whole of H by linearity and continuity. In a similar way, we may construct the mapping

$$w \to \langle\langle w, u_i\rangle_{H_i}, u_k\rangle_{H_k} \in \otimes_{r=1, r\neq i,k}^{n} H_r,$$

for $i \neq k$ and where u_i, u_k are fixed elements in H_i, H_k, respectively.

The analogue of Fubini's theorem holds, namely,

$$\langle\langle w, u_i\rangle_{H_i}, u_k\rangle_{H_k} = \langle\langle w, u_k\rangle_{H_k}, u_i\rangle_{H_i}.$$

In addition to the abstract problem (2.1), we introduce the operator equation

$$Bv = \sum_{j=1}^{n} \bar{\lambda}_j T_j v, \tag{4.1}$$

where B is a densely defined and closed linear operator in a separable Hilbert space h and the T_j, $1 \leqslant j \leqslant n$, are bounded symmetric operators in h. The n-tuple $(\lambda_1, \ldots, \lambda_n)$ is taken to be an eigenvalue of the problem (2.1).

The fundamental question now arises.

Does there exist a relationship between the solution v of (4.1) and the corresponding eigenvector $u^1 \otimes \cdots \otimes u^n$ of (2.1)?

To investigate this, we introduce the further tensor product space

$$\mathcal{H} = H \otimes h = \left(\otimes_{i=1}^{n} H_i\right) \otimes h$$

and define in $\mathcal{H}$ the determinantal operator

$$A = \det \begin{vmatrix} -A_1^{++} & S_{11}^{++} & \cdots & S_{1n}^{++} \\ \vdots & \vdots & & \vdots \\ -A_n^{++} & S_{n1}^{++} & \cdots & S_{n,n}^{++} \\ -B^{++} & T_1^{++} & \cdots & T_n^{++} \end{vmatrix}, \tag{4.2}$$

where, for example, S_{ij}^{++} is the operator in $\mathcal{H}$ induced by S_{ij}. The domain $D(A)$ of A is taken to be the algebraic tensor product

$$\left(\otimes_{a\,i=1}^{\;n} D(A_i)\right) \otimes_a D(B) \subseteq \mathcal{H}. \tag{4.3}$$

With these constructions, the main result of this section is the abstract relation embodied in the following theorem.

Theorem 4.1. *Let $\mathcal{K} \in \mathcal{H}$ be an element in the null space of $\bar{A}$ (the closure of A) and let $u = u^1 \otimes \cdots \otimes u^n$ be an eigenvector of (2.1) corresponding to the eigenvalue $(\lambda_1, \ldots, \lambda_n)$. Then*

$$v = \langle S^{++}\mathcal{K}, u\rangle_H \in D(B)$$

is a solution of (4.1), possibly trivial, with the same eigenvalue $(\lambda_1, \ldots, \lambda_n)$. Here $S^{++}; \mathcal{H} \to \mathcal{H}$ is the operator in $\mathcal{H}$ induced by S.

For the proof of this result, see Källström and Sleeman [17] and Sleeman [26].

We now illustrate the application of theorem (4.1) with a number of examples.

Example 1 $(n = 1)$**.** The classical Sturm–Liouville eigenvalue problem.

Let $H_1 = L^2(0, 1)$ and $A_1 : D(A_1) \subset H_1 \to H_1$ be the Sturm–Liouville operator

$$A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x),$$

where

$$D(A_1) = \{u, u' \text{ absolutely continuous locally on } [0, 1], u(0) = u(1) = 0\}.$$

For the operator $S_{11} : H_1 \to H_1$ we take

$$S_{11} = r(x)u.$$

In other words we are considering the eigenvalue problem

$$\begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (\lambda r(x) - q(x))u = 0, \qquad 0 < x < 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{4.4}$$

Now choose the space $h = H_1 = L^2(0, 1)$ and identify B with A_1 and T_1 with S_{11}. Then on $\mathcal{H} = H_1 \otimes H_2 = L^2(0, 1) \otimes L^2(0, 1)$, we define the determinantal operator

$$A = \begin{vmatrix} \frac{\partial^2}{\partial x^2} - q(x) & r(x) \\ \frac{\partial^2}{\partial y^2} - q(y) & r(y) \end{vmatrix},$$

which is none other than the partial differential operator

$$r(y)\left(\frac{\partial^2}{\partial x^2} - q(x)\right) - r(x)\left(\frac{\partial^2}{\partial y^2} - q(y)\right) \tag{4.5}$$

which is self-adjoint and defined on $D(A_1) \otimes_a D(A_1) \subseteq \mathcal{H}$.

Let $\mathcal{K}(x, y)$ be an element in the null space of (4.5), then theorem 4.1 shows that v must be a multiple of the eigenfunction u and that u satisfies the integral equation

$$u(y) = \mu \int_0^1 \mathcal{K}(x, y)r(x)u(x)\,\mathrm{d}x. \tag{4.6}$$

This is a classic result in the theory of the Sturm–Liouville problem in which the kernel $\mathcal{K}(x, y)$ turns out to be the Green function associated with (4.4). See, for example, Ince [13] or Hellwig [12].

Example 2a $(n = 2)$**.** Arscott's [1] two-parameter problem.

Here we investigate the two-parameter case of problem (1.1) under the positivity condition (A) in (1.3). Throughout we assume the Sturm–Liouville structure as in example 1.

Let

$$A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_1(x), \qquad A_2 = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + q_2(y),$$

S_{1j} represent multiplication by the continuous function $r_{1j}(x)$, $j = 1, 2, 0 \leqslant x \leqslant 1$ and S_{2j} represent multiplication by the continuous function $r_{2j}(y)$, $j = 1, 2, 0 \leqslant y \leqslant 1$. Here, $H_i = L^2(0, 1)$, $i = 1, 2$, and we assume the boundary conditions (1.2) are given.

Now choose $h = L^2(0, 1)$ and identify the operator B with A_1 and the operators T_i, $i = 1, 2$, with S_{1i}, $j = 1, 2$. That is,

$$B = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + q_1(z), \qquad T_1 = r_{11}(z), \qquad T_2 = r_{12}(z), \qquad 0 \leqslant z \leqslant 1.$$

The next step is to construct the determinantal operator

$$A = \det \begin{vmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - q_1(x) & r_{11}(x) & r_{12}(x) \\ \frac{\mathrm{d}^2}{\mathrm{d}y^2} - q_2(y) & r_{21}(y) & r_{22}(y) \\ \frac{\mathrm{d}^2}{\mathrm{d}z^2} - q_1(z) & r_{11}(z) & r_{12}(z) \end{vmatrix}.$$

When suitably interpreted, this is the partial differential operator

$$\begin{aligned} &(r_{11}(x)r_{22}(y) - r_{12}(x)r_{21}(y))\left(\frac{\partial^2}{\partial z^2} - q_1(z)\right) \\ &+ (r_{11}(z)r_{12}(x) - r_{11}(x)r_{12}(z))\left(\frac{\partial^2}{\partial y^2} - q_2(y)\right) \\ &+ (r_{12}(z)r_{21}(y) - r_{11}(z)r_{22}(y))\left(\frac{\partial^2}{\partial x^2} - q_1(x)\right). \end{aligned} \tag{4.7}$$

Now let $\mathcal{K}(x, y, z)$ be a suitably chosen element in the null space of (4.7), then from theorem (4.1) we can represent the eigenfunction $u_1(z)$ in terms of the eigenvector $u_1(x)u_2(y)$ via the integral relation

$$u_1(z) = \mu \int_0^1 \int_0^1 \mathcal{K}(x, y, z,)u_1(x)u_2(y)(r_{11}(x)r_{22}(y) - r_{12}(x)r_{21}(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.8}$$

Realization of this result in application to Lamé functions and ellipsoidal wavefunctions and other special functions are well known in the literature. See, for example, Arscott [2] and Browne and Sleeman [8] and the references cited therein.

Example 2b ($n = 2$)**.**

To introduce this example we recall the operator $\Delta_i : H \to H, i = 1, 2$ defined by the determinantal operator (2.10) and note that under condition (A), Δ_0 is strictly positive. In addition, we make use of the operators $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1, 2$ defined in section 3. These operators are pairwise commutative and fundamental to the completeness and expansion theory. For our purposes, it is enough to observe that if $u = u_1 \otimes u_2$ is an eigenvector of (2.1) with corresponding eigenvalue (λ_1, λ_2) then u is simultaneously an eigenvector of the set of problems

$$\Delta_1 u = \lambda_1 \Delta_0 u, \qquad \Delta_2 u = \lambda_2 \Delta_0 u. \tag{4.9}$$

Now, as in example 2a, we take

$$\begin{aligned} &A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_1(x), \qquad A_2 = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + q_2(y), \\ &S_{1j} = r_{1j}(x), \qquad j = 1, 2, \quad 0 \leqslant x \leqslant 1, \\ &S_{2j} = r_{2j}(y), \qquad j = 1, 2, \quad 0 \leqslant y \leqslant 1. \end{aligned}$$

Again $H_i = L^2(0,1), i = 1,2$ and the boundary conditions (1.2) are given. Now choose $h = H = L^2(0,1) \otimes L^2(0,1)$ and take the operator $B = \bar{\Delta}_1, T_1 = \Delta_0, T_2 = \phi$.

Note that the realization of B is the partial differential operator

$$\begin{aligned} B &= A_1 \otimes S_{22} - S_{12} \otimes A_2 \\ &= r_{22}(w)\left(-\frac{\partial^2}{\partial z^2} + q_1(z)\right) - r_{12}(z)\left(-\frac{\partial^2}{\partial w^2} + q_2(w)\right). \end{aligned} \tag{4.10}$$

The realization of the operator T_1 is the multiplication by the continuous function

$$T_1 = r_{11}(z)r_{22}(w) - r_{12}(z)r_{21}(w). \tag{4.11}$$

We now construct the determinantal operator

$$A = \det \begin{vmatrix} \frac{\partial^2}{\partial x^2} - q_1(x) & r_{11}(x) & r_{12}(x) \\ \frac{\partial^2}{\partial y^2} - q_2(y) & r_{21}(y) & r_{2,2}(y) \\ r_{22}(w)\left(\frac{\partial^2}{\partial z^2} - q_1(z)\right) - r_{12}(z)\left(\frac{\partial^2}{\partial w^2} - q_2(w)\right) & r_{11}(z)r_{22}(w) - r_{12}(z)r_{21}(w) & 0 \end{vmatrix},$$

which when written in full is the partial differential operator

$$\begin{aligned} A = {} & (r_{11}(x)r_{22}(y) - r_{12}(x)r_{21}(y))\left\{r_{22}(w)\left(\frac{\partial^2}{\partial z^2} - q_1(z)\right) - r_{12}(z)\left(\frac{\partial^2}{\partial w^2} - q_2(w)\right)\right\} \\ & - (r_{11}(z)r_{22}(w) - r_{12}(z)r_{21}(w)\left\{r_{22}(y)\left(\frac{\partial^2}{\partial x^2} - q_1(x)\right) - r_{12}(x)\left(\frac{\partial^2}{\partial y^2} - q_2(y)\right)\right\}. \end{aligned} \tag{4.12}$$

If $\mathcal{K}(x,y;z,w)$ is a suitably chosen element in the null space of (4.12), then from theorem (4.1) the eigenfunction $u_1(x)u_2(y)$ satisfies the integral equation

$$u_1(x)u_2(y) = \mu \int_0^1 \int_0^1 \mathcal{K}(x,y;z,w)u_1(z)u_2(w)\Delta_0(z,w)\,\mathrm{d}z\,\mathrm{d}w. \tag{4.13}$$

This integral equation, like the integral representation (4.8), has been realized for a number of the higher special functions such as Lamé functions and ellipsoidal wavefunctions. See Arscott [2], Browne and Sleeman [8] and Sleeman [24].

Example 2c $(n = 2)$**.**

This example, we believe, is new to the literature and views the two-parameter problem (1.1) as essentially a one-parameter problem with one spectral parameter held fixed. Let

$$A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_1(x) - \lambda_1 r_{11}(x), \qquad S_{11} = r_{12}.$$

Now choose $B : H \to H$ to be the partial differential operator (see (2.10))

$$\Delta_2 = r_{21}(z)\left(\frac{\partial^2}{\partial y^2} - q_1(y)\right) - r_{11}(y)\left(\frac{\partial^2}{\partial z^2} - q_2(z)\right) \tag{4.14}$$

and let the operator T_1 be the operation of multiplication by the continuous function

$$T_1 = r_{11}(y)r_{22}(z) - r_{12}(y)r_{21}(z) \equiv \Delta_0(y,z).$$

This time we construct the determinantal operator

$$A = \det \begin{vmatrix} \frac{\partial^2}{\partial x^2} - q_1(x) + \lambda_1 r_{11}(x) & r_{12}(x) \\ r_{21}(z)\left(\frac{\partial^2}{\partial y^2} - q_1(y)\right) - r_{11}(y)\left(\frac{\partial^2}{\partial z^2} - q_2(z)\right) & r_{11}(y)r_{22}(z) - r_{12}(y)r_{21}(z) \end{vmatrix},$$

which can be written as the partial differential operator

$$\begin{aligned}A = \Delta_0(y,z)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - q_1(x) + \lambda_1 r_{11}(x)\right)\\ - r_{12}(x)r_{21}(z)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - q_1(y) + \lambda_1 r_{11}(y)\right)\\ + r_{12}(x)r_{11}(y)\left(\frac{\partial^2}{\partial z^2} - q_2(z) + \lambda_1 r_{21}(z)\right).\end{aligned} \tag{4.15}$$

Now if $\mathcal{K}(x, y, z)$ is a suitably chosen element in the null space of (4.15) then from theorem (4.1)

$$u_1(y)u_2(z) = \mu \int_0^1 \mathcal{K}(x, y, z)u_1(x)r_{12}(x)\,\mathrm{d}x. \tag{4.16}$$

This integral relation appears to be new. However, a special form in application to ellipsoidal harmonics appears in Whittaker and Watson [30].

Before leaving this topic, it is appropriate to point out that the power of the abstract relation in theorem (4.1) depends in a crucial way on finding kernels $\mathcal{K}$ in the null space of the operator A in (4.2). That is, in application to Sturm–Liouville multiparameter problems, one needs to be able to solve in a non-trivial manner any one of the partial differential operators (4.7), (4.11) and (4.14). This is not a straightforward task but fortunately for many special functions posed as the solution to multiparameter problems a wide variety of kernels are known.

The result of example 2c is related to the work of Kuznetsov and Sklyanin [21].

5. Reduction and factorization of Schur polynomials

The subject of this section is motivated by the so-called Q-operator method used to solve a class of quantum integrable systems, see Kuznetsov *et al* [20] and the references cited therein.

Consider a quantum integrable system defined by n commuting linear partial differential operators L_i in n variables whose common eigenfunctions $\Psi_\lambda(\mathbf{x}) \equiv \Psi_\lambda(x_1, \ldots, x_n)$, i.e.

$$L_i\Psi_\lambda(\mathbf{x}) = h_i(\lambda)\Psi_\lambda(\mathbf{x})$$

form a basis in a Hilbert space $\mathcal{H}$. The multi-index $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a set of quantum numbers labelling the spectrum $h_i(\lambda)$ and the eigenfunctions $\Psi_\lambda(\mathbf{x})$.

By definition, a Q-operator Q_z depends on a parameter $z \in \mathbf{C}$ and satisfies the commutativity properties

$$[Q_{z_1}, Q_{z_2}] = 0 \quad \forall z_1, z_2 \in \mathbf{C}, \tag{5.1}$$

$$[Q_z, L_i] = 0 \quad \forall z \in \mathbf{C} \quad \text{and each} \quad i = 1, \ldots, n, \tag{5.2}$$

which imply that Q_z can be diagonalized by the basis functions $\Psi_\lambda(\mathbf{x})$, i.e.

$$[Q_z\Psi_\lambda](\mathbf{y}) = q_\lambda(z)\Psi_\lambda(\mathbf{y}). \tag{5.3}$$

A fundamental property of the Q-operator is that its eigenvalues $q_\lambda(z)$ satisfy a linear ordinary differential equation with respect to z, namely,

$$W\left(z, \frac{\mathrm{d}}{\mathrm{d}z}; \{h_i(\lambda)\}\right) q_\lambda(z) = 0, \tag{5.4}$$

whose coefficients depend on the eigenvalues $h_i(\lambda)$ of the commuting operators L_i. Equation (5.4) is called *Baxter's equation* or the *separation equation*. This equation will appear in a somewhat different guise in the discussion to follow.

A *separating operator* $\mathcal{S}_n$ factorizes the basis functions $\Psi_\lambda(\mathbf{x})$ into functions of one variable:

$$\mathcal{S}_n : \Psi_\lambda(x_1, \ldots, x_n) \to c_\lambda \prod_{i=1}^{n} \phi_\lambda^{(i)}(z_i), \tag{5.5}$$

where c_λ is a normalization constant. The functions $\phi_\lambda^{(i)}(z_i)$ are called *separated functions*.

In [21], Kuznetsov and Sklyanin observed that any Q-operator gives rise to a family of separating operators for an integrable system. That is, given a Q-operator Q_z and a linear functional ρ on $\mathcal{H}$ one can construct a product of n such operators

$$Q_{\mathbf{z}} = Q_{z_1} \cdots Q_{z_n} \tag{5.6}$$

and an integral operator $\mathcal{S}_n^{(\rho)} = \rho Q_{\mathbf{z}}$. It then follows that $\mathcal{S}_n^{(\rho)}$ is a family of separating operators parameterized by the functional ρ. By a suitable choice of the functional ρ one can simplify the structure of the integral operator $\mathcal{S}_n^{(\rho)}$. The principle goal of [20] is to study classes of Q-operators and the functional ρ for a variety of quantum integrable systems in order to construct the simplest separating operators $\mathcal{S}_n$ which factorize special functions $\Psi_\lambda(x_1, \ldots, x_n)$. In particular, Kuznetsov *et al* [20] make an in-depth study of the case when the special functions $\Psi_\lambda(x_1, \ldots, x_n)$ are Jack's [14] symmetric polynomials and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition.

In this section, we outline a new approach to the factorization problem and, for simplicity, illustrate the ideas in relation to the Schur polynomials [10]. In particular, we construct differential equations satisfied by Schur polynomials and then invoke various forms of the abstract relations discussed in section 4 to develop integral relations which lead to the desired factorization.

Schur polynomials are defined as

$$\begin{aligned} S_\lambda(x_1, \ldots, x_n) &= \frac{\det \begin{vmatrix} x_1^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ \vdots & & \vdots \\ x_1^{\lambda_{n-1}+1} & \cdots & x_n^{\lambda_{n-1}+1} \\ x_1^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{vmatrix}}{\det \begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{vmatrix}} \\ &\equiv \frac{g_\lambda(x_1, \ldots, x_n)}{g_0(x_1, \ldots, x_n)}, \end{aligned} \tag{5.7}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, is a partition of arbitrary weight $|\lambda|$,

$$|\lambda| = \sum_{i=1}^{n} \lambda_i.$$

The first step is to determine the partial differential equation satisfied by

$S_\lambda(x_1, \ldots, x_k, 1 \ldots, 1)$, $1 \leqslant k \leqslant n$. Observe that $g_\lambda(x_1, \ldots, x_n)$, considered as a function of the single variable x_i, satisfies the homogeneous ordinary differential equation

$$L(x_i) y \equiv \prod_{j=1}^{n} \left(x_i \frac{\mathrm{d}}{\mathrm{d}x_i} - (\lambda_j + n - j) \right) y = 0. \tag{5.8}$$

Consequently, it follows that $S_\lambda(x_1,\ldots,x_n)$ satisfies the partial differential equation

$$\sum_{i=1}^{n} L(x_i)g_0(x_1,\ldots,x_n)S_\lambda(x_1,\ldots,x_n)=0 \tag{5.9}$$

and $S_\lambda(x_1,\ldots,x_{n-1},1)$ satisfies the partial differential equation

$$\sum_{i=1}^{n-1} L(x_i)g_0(x_1,\ldots,x_{n-1},1)S_\lambda(x_1,\ldots,x_{n-1},1)=0. \tag{5.10}$$

By introducing the notation

$$\partial^r_{x_k} f \equiv \frac{\partial^r f}{\partial x_k^r}(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n), \qquad x_k=1$$

and successive use of L'Hospital's rule shows that $S_\lambda(x_1,\ldots,x_k,1,\ldots,1)$ satisfies the partial differential equation

$$\sum_{i=1}^{k} L(x_i)\left(\partial^{n-k-1}_{x_{k+1}}\partial^{n-k-2}_{x_{k+2}}\cdots\partial_{x_{n-1}}g_0(x_1,\ldots,x_k,1,\ldots,1)\right)S_\lambda(x_1,\ldots,x_k,1,\ldots,1)=0. \tag{5.11}$$

We now describe a suitable Hilbert space setting for the above partial differential expressions.

Let $\mathbb{C}(a,b,c,\ldots)$ be the field of rational functions in indeterminants $a,b,c,\ldots$ over the field $\mathbb{C}$ of complex numbers, $\mathbb{C}(a,b,c,\ldots)[x,y,z,\ldots]$, the ring of polynomials in the variables $x,y,z,\ldots$ with coefficients from $\mathbb{C}(a,b,c,\ldots)$ and $\mathbb{C}[x,y,z,\ldots]^{S_n}$ be the subring of symmetric polynomials. The Schur polynomials form a basis in $\mathbb{C}[\mathbf{x}]^{S_n}$. In the usual way, $\mathbb{C}[\mathbf{x}]^{S_n}$ can be completed to a Hilbert space when endowed with a suitable inner product. To begin with, we introduce the scalar product

$$\langle u,v\rangle=\int_{\mathcal{S}_{\mathbf{x}}} u(\mathbf{x})\overline{v(\mathbf{x})}\frac{(g_0(\mathbf{x}))^2}{\left(\prod_{i=1}^n x_i\right)^{n-1}}\,\mathrm{d}\mathbf{x} \tag{5.12}$$

where the domain of integration, by symmetry,

$$\mathcal{S}_{\mathbf{x}}=\{x_i=\exp 2\mathrm{i}q_i, i=1,\ldots,n,\mathbf{q}\in\mathbb{R}^n/\pi\mathbb{Z}^n\}. \tag{5.13}$$

The Schur polynomials are orthogonal with respect to the inner product (5.12). By completing $\mathbb{C}[\mathbf{x}]^{S_n}$ with respect to the inner product (5.12), we obtain a Hilbert space H_n. It is then a straightforward matter to show that the partial differential operator defined in (5.9) is symmetric on the space spanned by the Schur polynomials.

Following a similar line of reasoning, we can construct a suitable Hilbert space setting for the partial differential operator (5.10) defined on the space of polynomials $S_\lambda(x_1,\ldots,x_{n-1},1)$. These polynomials are orthogonal with respect to the inner product

$$\langle u,v\rangle=\int_{\mathcal{S}_{\mathbf{x},x_n=1}} u(\mathbf{x})\overline{v(\mathbf{x})}\frac{\left(g_0(\mathbf{x},x_n=1)\right)^2}{\left(\prod_{i=1}^{n-1} x_i\right)^{n-1}}\,\mathrm{d}\mathbf{x}. \tag{5.14}$$

We denote the resulting Hilbert space by H_{n-1}. The appropriate Hilbert space setting for the partial differential operator (5.11) is the inner product space spanned by the polynomials $S_\lambda(x_1,\ldots,x_k,1,\ldots,1)$ in which the following inner product is defined:

$$\langle u,v\rangle=\int_{\mathcal{S}_{(x_1,\ldots,x_k,1,\ldots,1)}} u(\mathbf{x})\overline{v(\mathbf{x})}\frac{\left(\partial^{n-k-1}_{x_{k+1}}\partial^{n-k-2}_{x_{k+2}}\cdots\partial_{x_{n-1}}g_0(x_1,\ldots,x_k,1,\ldots,1)\right)^2}{\left(\prod_{i=1}^{k} x_i\right)^{n-1}}\,\mathrm{d}\mathbf{x}. \tag{5.15}$$

The Hilbert space so constructed is denoted by H_k.

With these constructions, we are in a position to sketch our factorization programme. To begin with, we modify the notation defining the operators L_i and define

$$L(x_i) \equiv M(x_i) - \Omega_n, \tag{5.16}$$

where

$$\Omega_n \equiv \prod_{j=1}^{n} (\lambda_j + n - j). \tag{5.17}$$

We now wish to construct a map $A_1 : H_n \to H_{n-1}$ such that

$$A_1 g_0(x_1, \ldots, x_n) S_\lambda(x_1, \ldots, x_n) = g_0(x_1, \ldots, x_{n-1}, 1) S_\lambda(x_1, \ldots, x_{n-1}, 1). \tag{5.18}$$

To do this, we appeal to the abstract theory developed in section 4 and construct the tensor product space

$$\mathcal{H}_1 = H_n \otimes H_{n-1}. \tag{5.19}$$

Now let $\mathcal{K}(\mathbf{x}, \mathbf{y}; y_n = 1)$ be an appropriate element in the null space of the partial differential operator defined by the determinant

$$\det \begin{vmatrix} \sum_{i=1}^{n} M(x_i) & -nI_n \\ \sum_{i=1}^{n-1} M(y_i) & -(n-1)I_{n-1} \end{vmatrix} \tag{5.20}$$

then from theorem 4.1 we have the integral relation

$$g_0(x_1, \ldots, x_{n-1}, 1) S_\lambda(x_1, \ldots, x_{n-1}, 1) = \mu \int_{\mathcal{S}_\mathbf{t}} \mathcal{K}(\mathbf{t}, \mathbf{x}; x_n = 1) g_0(\mathbf{t}) S_\lambda(\mathbf{t})\, \mathrm{d}\mathbf{t}. \tag{5.21}$$

In general, we wish to construct a sequence of maps $A_{n-k+1} : H_k \to H_{k-1}, k = 1, \ldots, n$, such that

$$\begin{aligned} A_{n-k+1} &\left(\partial_{x_{k+1}}^{n-k-1} \partial_{x_{k+2}}^{n-k-2} \cdots \partial_{x_{n-1}} g_0(x_1, \ldots, x_k, 1, \ldots, 1)\right) S_\lambda(x_1, \ldots, x_k, 1, \ldots, 1) \\ &= \partial_{x_k}^{n-k-2} \partial_{x_{k+1}}^{n-k-3} \cdots \partial_{x_{n-1}} g_0(x_1, \ldots, x_{k-1}, 1, \ldots, 1) S_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1). \end{aligned} \tag{5.22}$$

Now let $\mathcal{K}(x_1, \ldots, x_k, y_1, \ldots, y_{k-1})$ be an appropriate element in the null space of the partial differential operator defined by the determinant

$$\det \begin{vmatrix} \sum_{i=1}^{k} M(x_i,) & -kI_k \\ \sum_{i=1}^{k-1} M(y_i,) & -(k-1)I_{k-1} \end{vmatrix}. \tag{5.23}$$

Then again from theorem 4.1, we have

$$\begin{aligned} &\partial_{x_k}^{n-k-2} \partial_{x_{k+1}}^{n-k-3} \cdots \partial_{x_{n-1}} g_0(x_1, \ldots, x_{k-1}, 1, \ldots, 1) S_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1) \\ &\quad = \mu \int_{\mathcal{S}_\mathbf{t}} \mathcal{K}(t_1, \ldots, t_k, y_1, \ldots, y_{k-1}) \left(\partial_{t_{k+1}}^{n-k-1} \partial_{t_{k+2}}^{n-k-2} \cdots \partial_{t_{n-1}} g_0(t_1, \ldots, x_k, 1, \ldots, 1)\right) \\ &\qquad \times S_\lambda(t_1, \ldots, t_k, 1, \ldots, 1)\, \mathrm{d}\mathbf{t}, \\ &\qquad k = 1, \ldots, n. \end{aligned} \tag{5.24}$$

The next task is to address the problem of finding a sequence of maps $B_{n-k+1} : H_k \to H_k$, $k = 1, \ldots, n$, such that

$$\begin{aligned} B_{n-k+1} &\left(\partial_{x_{k+1}}^{n-k-1} \partial_{x_{k+2}}^{n-k-2} \cdots \partial_{x_{n-1}} g_0(x_1, \ldots, x_k, 1, \ldots, 1)\right) S_\lambda(x_1, \ldots, x_k, 1, \ldots, 1) \\ &= \partial_{x_k}^{n-k-2} \partial_{x_{k+1}}^{n-k-3} \cdots \partial_{x_{n-1}} g_0(x_1, \ldots, x_{k-1}, 1, \ldots, 1) \\ &\quad \times S_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1) q_\lambda(z), \end{aligned} \tag{5.25}$$

where $q_\lambda(z)$ is an eigenfunction of a certain ordinary differential operator. Our abstract theory allows one to employ almost any such differential operator. However, a study of the work of Kuznetsov *et al* [20] suggests $q_\lambda(z)$ be proportional to Fox's hypergeometric function [10] and satisfies

$$L(z)q_\lambda(z) \equiv M(z)q_\lambda(z) - \Omega_n q_\lambda(z). \tag{5.26}$$

With this choice of eigenfunction we now carry a programme similar to the above. That let $\mathcal{K}(\mathbf{x}, \mathbf{y}; y_n = 1, z)$ be an appropriate element in the null space of

$$\det \begin{vmatrix} \sum_{i=1}^{n} M(x_i) & -nI_n \\ \sum_{i=1}^{n-1} M(y_i) + M(z) & -nI_n \end{vmatrix}. \tag{5.27}$$

Then theorem 4.1 shows that the following integral relation holds:

$$g_0(x_1, \ldots, x_{n-1}, 1)S_\lambda(x_1, \ldots, x_{n-1}, 1)q_\lambda(z) = \mu \int_{\mathcal{S}_\mathbf{t}} \mathcal{K}(\mathbf{t}, \mathbf{x}; x_n = 1, z)g_0(\mathbf{t})S_\lambda(\mathbf{t})\,\mathrm{d}\mathbf{t}. \tag{5.28}$$

In a similar fashion, we may also construct the integral relation

$$\begin{aligned} &\partial_{x_k}^{n-k-2}\partial_{x_{k+1}}^{n-k-3}\cdots\partial_{x_{n-1}} g_0(x_1, \ldots, x_{k-1}, 1, \ldots, 1)S_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1)q_\lambda(z) \\ &\quad = \mu \int_{\mathcal{S}_\mathbf{t}} \mathcal{K}(t_1, \ldots, t_k, y_1, \ldots, y_{k-1}, z)\partial_{t_{k+1}}^{n-k-1}\partial_{t_{k+2}}^{n-k-2}\cdots\partial_{t_{n-1}} g_0 \\ &\quad \times (t_1, \ldots, x_k, 1, \ldots, 1)S_\lambda(t_1, \ldots, t_k, 1, \ldots, 1)\,\mathrm{d}\mathbf{t}, \end{aligned} \tag{5.29}$$

where this time we seek a kernel $\mathcal{K}$ satisfying the partial differential equation realization of

$$\det \begin{vmatrix} \sum_{i=1}^{k} M(x_i,) & -kI_k \\ \sum_{i=1}^{k-1} M(y_i,) + M(z) & -kI_k \end{vmatrix}. \tag{5.30}$$

These integral relations define the desired maps $B_{n-k+1} : H_k \to H_k, k = 1, \ldots, n$. It is instructive to briefly construct some solutions of the partial differential equation (5.30) which when written in full is

$$\left[\sum_{i=1}^{k} M(x_i) - \sum_{i=1}^{k-1} M(y_i) - M(z)\right]\mathcal{K} = 0, \tag{5.31}$$

that is

$$\begin{aligned} &\sum_{i=1}^{k}\prod_{j=1}^{n}\left(x_i\frac{\partial}{\partial x_i} - (\lambda_j + n - j)\right)\mathcal{K} - \sum_{i=1}^{k-1}\prod_{j=1}^{n}\left(y_i\frac{\partial}{\partial y_i} - (\lambda_j + n - j)\right)\mathcal{K} \\ &\quad = \prod_{j=1}^{n}\left(z\frac{\partial}{\partial z} - (\lambda_j + n - j)\right)\mathcal{K}. \end{aligned} \tag{5.32}$$

One wide class of solutions can be obtained in the following manner. First, it is not difficult to show, see [10, 20], that (5.8) has the polynomial solution

$$G_\lambda(x) = x^{\lambda_n}(1-x)^{1-n}{}_nF_{n-1}\left(\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ & & & & ; x \\ b_1, & b_2, & \ldots, & b_{n-1} \end{matrix}\right),$$

where

$$a_i = (\lambda_n - n) - (\lambda_i - i), \quad i = 1, \ldots, n \quad \text{and} \quad b_j = a_j + 1, \quad j = 1, \ldots, n-1.$$

We now introduce the notation

$$\mathbf{x}^k \equiv \prod_{i=1}^{k} x_i, \qquad k = 1, \ldots, n. \tag{5.33}$$

An appropriate kernel $\mathcal{K}$ satisfying (5.32) is

$$\mathcal{K}(x_1, \ldots, x_k, y_1, \ldots, y_{k-1}, z) \\ = (\mathbf{x}^k\mathbf{y}^{k-1}z)^{\lambda_n}(1-\mathbf{x}^k\mathbf{y}^{k-1}z)^{1-n}{}_nF_{n-1}\begin{pmatrix} a_1, & a_2, & \ldots, & a_n & \\ & & & & ;\mathbf{x}^k\mathbf{y}^{k-1}z \\ b_1, & b_2, & \ldots, & b_{n-1} & \end{pmatrix}. \tag{5.34}$$

In order to see how (5.34) is derived and at the same time to see how a wide class of kernel solutions to (5.32) can be obtained, we make use of the transformation implied in the definition (5.13). That is, we set

$$x_j = \exp 2\mathrm{i}q_j, \qquad y_j = \exp 2\mathrm{i}r_j, \qquad z = \exp 2\mathrm{i}t$$

and then (5.32) takes the form

$$\sum_{i=1}^{k}\prod_{j=1}^{n}\left(\frac{\partial}{\partial q_i} - 2\mathrm{i}(\lambda_j + n - j)\right)\mathcal{K} - \sum_{i=1}^{k-1}\prod_{j=1}^{n}\left(\frac{\partial}{\partial r_i} - 2\mathrm{i}(\lambda_j + n - j)\right)\mathcal{K} \\ = \prod_{j=1}^{n}\left(\frac{\partial}{\partial t} - 2\mathrm{i}(\lambda_j + n - j)\right)\mathcal{K}. \tag{5.35}$$

This is a partial differential equation with constant coefficients and furthermore due to its symmetric form is formally satisfied by any function of the form

$$K(q_1, \ldots, q_k; r_1, \ldots, r_{k-1}; t) = F\left(\sum_{i=1}^{k} q_i + \sum_{i=1}^{k-1} r_i + t\right). \tag{5.36}$$

Indeed, F could be chosen to be a polynomial of its argument. In particular, if we transform back to the original variables one such polynomial is precisely (5.34).

A particularly important observation is that F need not depend on the parameters λ_i and this opens the possibility of generating an elementary kernel which is essentially independent of the factorization (5.29). That is, it should be possible to find a fixed kernel appropriate for each step of the factorization. This is a key point in the work of Kuznetsov *et al* [20].

6. Open problems and new directions

Historically, multiparameter problems arose out of applying the method of separation of variables to partial differential equations of physics and engineering. It has therefore concentrated on applications to boundary-value or eigenvalue problems for ordinary differential equations, namely, the multiparameter Sturm–Liouville problem. However, there is still much to learn. In particular, it is of considerable importance to study the spectral properties of a multiparameter system when some or all of the linked differential equations are defined on unbounded intervals. In other words, what is the multiparameter analogue of the Weyl limit point, limit circle theory as discussed for example in [12]? A first attempt at such a theory is contained in [25]. As far as an abstract theory is concerned, Atkinson [4] has pioneered the finite-dimensional case of matrices. However, it would be of considerable interest to consider multiparameter problems for difference operators. This has important applications in mathematical physics. In another direction, it is important to note that the abstract formulation discussed in section 2 deals entirely with a Hilbert space setting. It is therefore natural to seek a development of the theory in a Banach Space. Furthermore, our formulation has consistently assumed that the operators S_{ij} are bounded symmetric operators in a Hilbert space H_i and that the A_i are self-adjoint operators in H_i. It would be of interest to develop the theory when the S_{ij} are not necessarily bounded. If the S_{ij} are unbounded but compact relative to A_i, and A_i is positive, we can reformulate the problem in terms of the identity operators I_i and the operators $A_i^{-1}S_{ij}$ and the theory applies.

The topic of section 2, namely solvability, is fundamental to the development of multiparameter spectral theory. It is intriguing that a number of commutativity identities involving the operators emerge. Although we have not exploited these properties they do suggest new ways of formulating multiparameter operator problems. In particular, the above commutativity results are intimately connected with the general abstract problem of the study of the joint spectra for several commuting operators as discussed by Taylor [28] and Källström and Sleeman [19]. Again there is a need to extend the solvability results to the situation when some or all of the S_{ij} are unbounded.

The abstract relations discussed in section 4 open up a new and novel way of studying multiparameter spectral theory. This has yet to be done in general. It is however clear that the ideas developed in section 4 provide a unified way of obtaining many integral equations and relations for special functions arising in physics and engineering. Furthermore, it also indicates a framework in which to study special functions which do not necessarily arise from the classical ideas of separation of variables. The subject of Schur polynomials considered in section 5 is an interesting example of the possible way in which to address the important factorization problems of interest to mathematical physics. It would be very interesting to explore these ideas in application to Jack polynomials [14–16] and possibly the Macdonald polynomials [22].

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